Some remarks on L-type mappings

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Chebyshev's radius of C:

$$r(C) := \inf\{r_x(C) : \ x \in B\}.$$

The same definitions work in the case of a bounded subset of a metric space.

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Lemma

B does not have normal structure if and only if it contains a **diametral sequence**, that is, the sequence (x_n) for which:

$$\lim \mathrm{dist}(x_{n+1}, \bar{co}\{x_1, \ldots, x_n\}) = \mathrm{diam}\{x_1, x_2, \ldots\}.$$

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$$\left|\, e_{n+1} - x \,\right|_R = \max\{\sqrt{1 + \|x\|_2^2}, R\}.$$

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Conclusion

An asymptotically nonexpansive mapping T has afps: $a_n = T^n(x),\ x \in C$.

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if

1. for each closed convex and T-invariant set $C_0 \subset C$: there is afps of T in C_0;
2. for each (a_n) – afps of T and each $x \in C$:

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Definition

$T \colon C \to C$ is called a quasi-nonexpansive mapping
if for each fixed point x_0:

$$\|x_0 - T(x)\| \leqslant \|x_0 - x\|, \qquad \forall x \in C_0.$$

Known results

Theorem (Llorens-Fuster, Moreno-Galvez, 2011)

Let B be a Banach space with a normal structure and $C \subset B$. Then each L-type nonexpansive mapping $T : C \to C$ has a fixed point.

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Theorem (Llorens-Fuster, Moreno-Galvez, 2011)

Let B be a Banach space with a normal structure and $C \subset B$. Then each L-type nonexpansive mapping $T : C \to C$ has a fixed point.

Claim (Llorens-Fuster, Moreno-Galvez, 2011)

There is a continuous quasi-nonexpansive mapping defined on a compact convex set, which is not of L-type.

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Revised claim

Each continuous quasi-nonexpansive mapping, defined on a compact convex set, is of L-type.

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Let $B_n = B(x_n, d/3) \cap K$ and $a_n \in B_n$:
$\|a_n - x_n\| < 1/n.$

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$$T(x) = \begin{cases} x_{n+1}, & x = B_n \setminus \{a_n\} \\ x_n, & x = a_n \\ x_1, & \text{otherwise} \end{cases} .$$

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- T is not continuous;
- T is of L-type;
- $Fix\{T\} = \emptyset$.

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YES, but the mapping is not asymptotically regular.

References

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Thank you very much
for your attention