Birkhoff-Kellogg type results in cones with applications

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A well known result in nonlinear analysis is the Birkhoff-Kellogg invariant-direction Theorem [4], cf. [16, Theorem 6.1].

Theorem 1 (Birkhoff-Kellogg)

Let U be a bounded open neighborhood of 0 in an infinite-dimensional normed linear space $(V, \| \, \|)$, and let $T : \partial U \to V$ be a compact map satisfying $\|T(x)\| \geq \alpha$ for some $\alpha > 0$ for every x in ∂U. Then there exist $x_0 \in \partial U$ and $\lambda_0 \in (0, +\infty)$ such that $x_0 = \lambda_0 T(x_0)$.

This theorem has been object of deep studies in the past, with applications and extensions in several directions, e.g. [2, 5, 10, 12, 13, 14, 15, 17, 24, 26, 30, 29] and references therein. In particular [12, 24, 30] provide interesting applications to the existence of eigenvalues and eigenfunctions of elliptic BVPs.

We use of the following result, which is set in cones.

A cone K of a real Banach space $(X, \| \, \|)$ is a closed set with $K + K \subset K$, $\mu K \subset K$ for all $\mu \geq 0$ and $K \cap (-K) = \{0\}$.

We set $K_r := \{x \in K : \|x\| < r\}$, $\overline{K}_r := \{x \in K : \|x\| \leq r\}$, $\partial K_r := \{x \in K : \|x\| = r\}$.

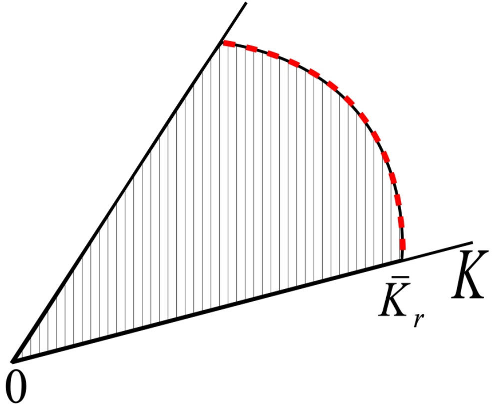
Theorem 2 (Krasnosel'skiĭ and Ladyženskiĭ)

Let $(X, \| \, \|)$ be a real Banach space, let $T : \overline{K}_r \to K$ be compact and suppose that

$$\inf_{x \in \partial K_r} \|Tx\| > 0.$$

Then there exist $\lambda_0 \in (0, +\infty)$ and $x_0 \in \partial K_r$ such that

$$x_0 = \lambda_0 T x_0.$$


$\bar{K}_r$
K
0

A classical application of Theorem 2 can be found in the book of Guo and Lakshmikantham [18] for the case of the nonlinear eigenvalue problem

$$u''(t) = \lambda f(u(t)),\ t \in (0,1);\ \ u(0) = u(1) = 0; \qquad (1)$$

the BVP(1) is rewritten as an eigenvalue problem in a suitable cone of positive functions in $C[0,1]$.

Theorem 2 has been utilized by GI in [19] to prove the existence of positive eigenvalues with associated eigenfunctions that are allowed to *change sign* for a set of three-point BVPs including a nonlocal version of (1), namely

$$u''(t) = \lambda f(u(t));\ u(0) = 0,\ \ u(1) = \alpha u(\eta),\ \eta \in (0,1). \qquad (2)$$

More recently, Theorem 2 has been used by GI [20] to study the existence of eigenvalues and positive eigenfunctions of the system of second order elliptic functional differential equations subject to functional boundary conditions

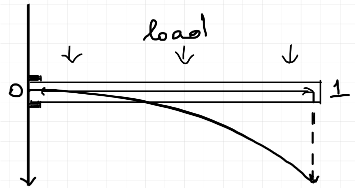
$$\begin{cases} L_i u_i = \lambda f_i(x, u, Du, w_i[u]), & \text{in } \Omega, \quad i = 1, 2, \dots, n, \\ B_i u_i = \lambda \zeta_i(x) h_i[u], & \text{on } \partial\Omega, \quad i = 1, 2, \dots, n, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary, L_i is a strongly uniformly elliptic operator, B_i is a first order boundary operator, $u = (u_1, \dots, u_n)$, $Du = (\nabla u_1, \dots, \nabla u_n)$, f_i are continuous functions, ζ_i are sufficiently regular functions, w_i and h_i are suitable compact functionals.

In the context of higher order equations of ODEs, Theorem 2 has been used by GI [20] in order to discuss the solvability of the parameter-dependent BVP

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t), u'(t), u''(t), u'''(t)), \ t \in (0,1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + \lambda H[u] = 0, \end{cases} \tag{4}$$

where f is a continuous function, H is a suitable compact functional in the space $C^3[0,1]$ and λ is a non-negative parameter. The BVP (4) can be used as a model for a cantilever bar.



When $H[u] \equiv 0$ it models a bar of length 1 is clamped on the left end and the right end is free to move with vanishing bending moment and shearing force, see for example [1, 27, 34].

Under a mechanical point of view, interesting cases appear when the shearing force at the right side of the beam does not vanish, see for example GI and Pietramala [22]:

- $u'''(1) + k_0 = 0$ models a force acting in 1,
- $u'''(1) + k_1 u(1) = 0$ describes a spring in 1,
- $u'''(1) + g(u(1)) = 0$ models a spring with a strongly nonlinear rigidity,
- $u'''(1) + g(u(\eta)) = 0$ describes a feedback mechanism, where the spring reacts to the displacement registered in a point η of the beam.

By means of critical point theory Cabada and Terzian [6] and Bonanno, Chinnì and Terzian in [3] studied the parameter-dependent BVP

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \ t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + \lambda g(u(1)) = 0. \end{cases} \tag{5}$$

Insofar as the generality of BCs is concerned, by classical fixed point index, Cianciaruso, GI and Pietramala in [9] studied the BVP

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \ t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + H(u) = 0, \end{cases} \tag{6}$$

where H is a suitable functional (non necessarily linear) on $C[0, 1]$.

Regarding higher order dependence, the ODE

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, 1]$$

under the homogeneous BCs

$$u(0) = u'(0) = u''(1) = u'''(1) = 0$$

has been studied by Li [27] via fixed point index, while the non-homogeneous case

$$u(0) = u'(0) = u''(1) = 0, u'''(1) + g(u(1)) = 0$$

has been studied by Wei, Li and Li [32] and the case

$$u(0) = u'(0) = \int_0^1 p(t)u(t)\,dt, u''(1) = u'''(1) = \int_0^1 q(t)u''(t)\,dt,$$

has been investigated by Khanfer and Bougoffa [23] via the Schauder fixed point theorem.

It is known (see for example Lemma 2.1 and Lemma 2.2 of [27]) that for $h \in C[0,1]$ the unique solution of the linear BVP

$$\begin{cases} u^{(4)}(t) = h(t), \; t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

is given by

$$u(t) = \int_0^1 k(t,s)h(s)\,ds, \tag{7}$$

where

$$k(t,s) = \begin{cases} \frac{1}{6}(3t^2 s - t^3), & s \geq t \\ \frac{1}{6}(3s^2 t - s^3), & s \leq t. \end{cases}$$

The Green's function k has the following properties

$$k(t,s), \frac{\partial k}{\partial t}(t,s), \frac{\partial^2 k}{\partial t^2}(t,s) \geq 0 \text{ on } [0,1]\times[0,1],$$

and

$$\frac{\partial^3 k}{\partial t^3}(t,s) \leq 0 \text{ on } [0,1]^2 \setminus \{(t,s) | t=s)\}.$$

Furthermore note that (see for example [22])

$$\gamma(t) = \frac{1}{6}(3t^2 - t^3),$$

is the unique solution of the BVP

$$\gamma^{(4)}(t) = 0,\ \gamma(0) = \gamma'(0) = \gamma''(1) = 0,\ \gamma'''(1) + 1 = 0.$$

By direct calculation, it can be observed that

$$\gamma(t), \gamma'(t), \gamma''(t), -\gamma'''(t) \geq 0 \text{ on } [0, 1].$$

We work in the space $C^3[0,1]$ endowed with the norm

$$\|u\|_3 := \max_{j=0,\dots,3}\{\|u^{(j)}\|_\infty\}, \text{ where } \|w\|_\infty = \sup_{t\in[0,1]} |w(t)|.$$

Definition 3

We say that λ is an eigenvalue of the BVP (4) with a corresponding eigenfunction $u \in C^3[0,1]$ with $\|u\|_3 > 0$ if the pair (u,λ) satisfies the perturbed Hammerstein integral equation

$$u(t) = \lambda\Big(\gamma(t)H[u] + \int_0^1 k(t,s)f(s,u(s),u'(s),u''(s),u'''(s))\,ds\Big). \tag{8}$$

We make use of the cone

$$K := \{u \in C^3[0,1] : \ u, u', u'', -u''' \geq 0, \ \text{for every } t \in [0,1]\}.$$

(a larger cone than the one used in [27]) and consider the sets

$$K_\rho := \{u \in K : \|u\|_3 < \rho\}, \overline{K}_\rho := \{u \in K : \|u\|_3 \leq \rho\},$$

$$\partial K_\rho := \{u \in K : \|u\|_3 = \rho\},$$

where $\rho \in (0, +\infty)$.

Theorem 4 (GI [21])

Let $\rho \in (0, +\infty)$ *and assume the following conditions hold.*

(a) $f \in C(\Pi_\rho, \mathbb{R})$ *and there exist* $\underline{\delta}_\rho \in C([0,1], \mathbb{R}_+)$ *such that*

$$f(t,u,v,w,z) \geq \underline{\delta}_\rho(t), \text{ for every } (t,u,v,w,z) \in \Pi_\rho,$$

where

$$\Pi_\rho : [0,1] \times [0,\rho]^3 \times [-\rho, 0].$$

(b) $H : \overline{K}_\rho \to \mathbb{R}_+$ *is continuous and bounded. Let* $\underline{\eta}_\rho \in [0, +\infty)$ *be such that*

$$H[u] \geq \underline{\eta}_\rho, \text{ for every } u \in \partial K_\rho.$$

Theorem (cont.)

(c) *The inequality*

$$\underline{\eta}_\rho + \int_0^1 \underline{\delta}_\rho(s)\, ds > 0 \tag{9}$$

holds.

Then the BVP (4) has a positive eigenvalue λ_ρ with an associated eigenfunction $u_\rho \in \partial K_\rho$

Corollary 5

In addition to the hypotheses of Theorem 4, assume that ρ can be chosen arbitrarily in $(0, +\infty)$. Then for every ρ there exists a non-negative eigenfunction $u_\rho \in \partial K_\rho$ of the BVP (4) to which corresponds a $\lambda_\rho \in (0, +\infty)$.

Theorem 6 (GI [21])

In addition to the hypotheses of Theorem 4 assume the following conditions hold.

(d) *There exist* $\overline{\delta}_\rho \in C([0,1],\mathbb{R}_+)$ *such that*

$$f(t,u,v,w,z) \leq \overline{\delta}_\rho(t), \text{ for every } (t,u,v,w,z) \in \Pi_\rho.$$

(e) *Let* $\overline{\eta}_\rho \in [0,+\infty)$ *be such that*

$$H[u] \leq \overline{\eta}_\rho, \text{ for every } u \in \partial K_\rho.$$

Then λ_ρ *satisfies the following estimates*

$$\frac{\rho}{\left(\overline{\eta}_\rho + \int_0^1 \overline{\delta}_\rho(s)\, ds\right)} \leq \lambda_\rho \leq \frac{\rho}{\left(\underline{\eta}_\rho + \int_0^1 \underline{\delta}_\rho(s)\, ds\right)}.$$

An example

Consider the BVP

$$\begin{cases} u^{(4)}(t) = \lambda t e^{u(t)}(1 + (u'''(t))^2), \ \ t \in (0,1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) + \lambda\Big(\frac{1}{1+(u(\frac{1}{2}))^2} + \int_0^1 t^3 u''(t)\, dt\Big) = 0. \end{cases} \tag{10}$$

Fix $\rho \in (0, +\infty)$ may take

$$\underline{\eta}_\rho(t) = \frac{1}{1+\rho^2}, \overline{\eta}_\rho(t) = 1 + \frac{\rho}{4},$$

$$\underline{\delta}_\rho(t) = t, \overline{\delta}_\rho(t) = te^{\rho}(1+\rho^2).$$

Thus we have

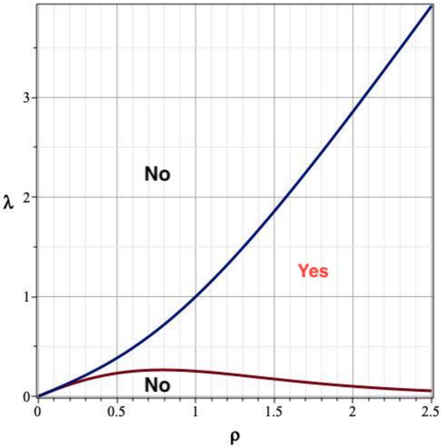
$$\underline{\eta}_\rho + \int_0^1 \underline{\delta}_\rho(s)\, ds = \frac{1}{1+\rho^2} + \int_0^1 s\, ds \geq \frac{1}{2},$$

which implies that (9) is satisfied for every $\rho \in (0, +\infty)$.

Thus we can apply Corollary 5 and Theorem 6, obtaining uncountably many pairs of positive eigenvalues and eigenfunctions (u_ρ, λ_ρ), where $\|u_\rho\|_3 = \|u_\rho'''\|_\infty = \rho$ and

$$\frac{4\rho}{2\mathrm{e}^\rho \rho^2 + 2\mathrm{e}^\rho + \rho + 4} \leq \lambda_\rho \leq \frac{2\rho\left(\rho^2+1\right)}{\rho^2+3}.$$

The next figure illustrates the localization of the (u_ρ, λ_ρ) pairs.



Translates of cones

Let $(X, \| \|)$ be a real Banach space and K be a cone in X. For $y \in X$, the *translate* of the cone K is defined as

$$K_y := y + K = \{y + x : x \in K\}.$$

Given an open bounded subset D of X we denote $D_{K_y} = D \cap K_y$, an open subset of K_y.

In this context, utilizing classical fixed point index, we have the following BK-type theorem.

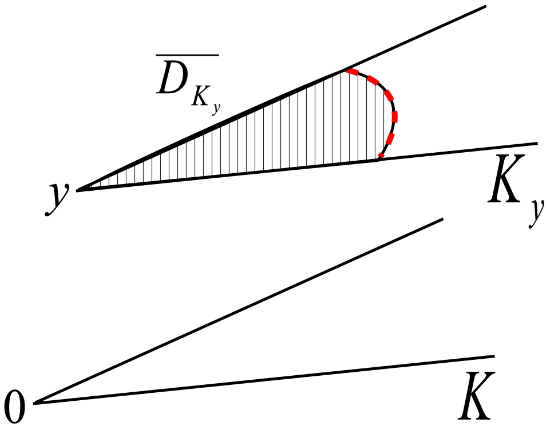
Theorem 7 (Calamai and GI [7])

Let $(X, \| \, \|)$ be a real Banach space, $K \subset X$ be a cone and $D \subset X$ be an open bounded set with $y \in D_{K_y}$ and $\overline{D}_{K_y} \neq K_y$. Assume that $\mathcal{F} : \overline{D}_{K_y} \to K$ is a compact map and assume that

$$\inf_{x \in \partial D_{K_y}} \|\mathcal{F}(x)\| > 0.$$

Then there exist $x^* \in \partial D_{K_y}$ and $\lambda^* \in (0, +\infty)$ such that

$$x^* = y + \lambda^* \mathcal{F}(x^*).$$


$\overline{D_{K_y}}$
y
K_y
0
K

Theorem 7 seems to be quite useful when dealing with problems with delay effects.

Third order functional differential equations with nonlocal boundary terms have been studied in the past, we mention here, for example, the work of Tsamatos [31] and the subsequent papers [11, 33, 28].

Recently Calamai and GI [7] discussed the solvability of the following set of third order parameter-dependent functional differential equations with functional BCs.

$$u'''(t) + \lambda F(t, u_t) = 0, \;\; t \in [0, 1], \tag{11}$$

with initial conditions

$$u(t) = \psi(t), \;\; t \in [-r, 0], \tag{12}$$

and one of the following BCs (here $B[\cdot]$ acts on $C^1([-r, 1], \mathbb{R})$)

$$u(0) = u'(0) = 0, \;\; u(1) = \lambda B[u], \tag{13}$$

$$u(0) = u'(0) = 0, \;\; u'(1) = \lambda B[u], \tag{14}$$

$$u(0) = u'(0) = 0, \;\; u''(1) = \lambda B[u]. \tag{15}$$

The idea in this case is to rewrite the problem as an integral equation in the space $C^1([-r,1],\mathbb{R})$:

$$u(t) = \psi(t) + \lambda\Big(\int_0^1 k(t,s)F(s,u_s)\,ds + \gamma(t)B[u]\Big) \\ =: \psi(t) + \lambda\mathcal{F}u(t), \quad t \in [-r,1], \quad (16)$$

and apply Theorem 7.

Theorem 8 (Calamai and GI [7])

Let $\rho \in (0, +\infty)$ and assume the following further conditions hold.

(a) *There exist $\underline{\delta}_\rho \in C([0,1], \mathbb{R}_+)$ such that*

$$F(t,\phi) \geq \underline{\delta}_\rho(t), \text{ for every } (t,\phi) \in [0,1] \times C^1([-r,0], \mathbb{R}) \text{ with } \|\phi\|_{[-r,0],1} \leq \max\{\rho, \|\psi\|_{[-r,1],1}\}.$$

(b) *$B : \overline{K}_{\psi,\rho} \to \mathbb{R}_+$ is continuous and bounded. Let $\underline{\eta}_\rho \in [0, +\infty)$ be such that*

$$B[u] \geq \underline{\eta}_\rho, \text{ for every } u \in \partial K_{\psi,\rho}.$$

Theorem (cont.)

(c) *The inequality*

$$\sup_{t\in[0,1]}\left\{\gamma(t)\underline{\eta}_{\rho}+\int_{0}^{1}k(t,s)\underline{\delta}_{\rho}(s)\,ds\right\}>0 \tag{17}$$

holds.

Then there exist λ_ρ *and* $u_\rho \in \partial K_{\psi,\rho}$ *such that the integral equation* (16) *is satisfied.*

A second example

We adapt the nonlinearities studied in Example 2.6 of [21] to this context by considering the family of FBVPs

$$u'''(t)+\lambda t e^{u(t)+\left(u'\left(t-\frac{1}{2}\right)\right)^2}\left(1+(u'(t))^2+\left(u\left(t-\frac{1}{3}\right)\right)^2\right)=0,\ t\in(0,1), \tag{18}$$

with the initial condition

$$u(t)=\psi(t), t\in\left[-\frac{1}{2},0\right], \tag{19}$$

with $\psi(t)=H(-t)t^2$, and one of the three BCs (13), (14), (15), where we fix

$$B[u]=\frac{1}{1+(u(\frac{1}{2}))^2}+\int_{-\frac{1}{2}}^{1} t^3(u'(t))^2\,dt.$$

Now choose $\rho \in (0, +\infty)$. Thus we may take

$$\underline{\eta}_\rho(t) = \frac{1}{1+\rho^2}, \underline{\delta}_\rho(t) = t.$$

Therefore, for every $i = 1, 2, 3$, we have

$$\sup_{t\in[0,1]} \left\{ \frac{\gamma_i(t)}{1+\rho^2} + \int_0^1 k_i(t,s) t\, ds \right\} \geq \frac{1}{2(1+\rho^2)} > 0,$$

which implies that (17) is satisfied for every $\rho \in (0, +\infty)$.

Thus we can apply Theorem 8, obtaining uncountably many pairs of solutions and parameters (u_ρ, λ_ρ) for the FBVPs (18)-(19)-(13), (18)-(19)-(14) and (18)-(19)-(15).

Fourth order case

The affine BK-type theorem has been used by Calamai and GI [8] for the following class of BVPs:

$$u^{(4)}(t) + \lambda F(t, u_t) = 0, \;\; t \in [0,1], \tag{20}$$

with initial conditions

$$u(t) = \psi(t), \;\; t \in [-r, 0], \tag{21}$$

and one of the following BCs

$$u^{(j)}(1) = \lambda B[u], \tag{22}$$

where j can be either 0 or 1, 2, 3.

In a similar way as above we seek solutions of the integral equation

$$u(t)=\widehat{\psi}(t)+\lambda\Big(\int_0^1 k(t,s)H(t)F(s,u_s)\,ds+H(t)\gamma(t)B[u]\Big),\ \ t\in[-r,1], \tag{23}$$

by using the affine cone

$$K_{\widehat{\psi}}=\widehat{\psi}+K_0,$$

where

$$\begin{aligned}K_0=\{u\in C^2([-r,1],\mathbb{R}):\, & u(t)\geq 0\ \forall t\in[-r,1],\\ & \text{and } u(t)=u'(t)=u''(t)=0\ \forall t\in[-r,0]\}.\end{aligned}$$

A result similar to Theorem 8 holds in this case.

A third example

We consider the family of FBVPs

$$u^{(4)}(t)+\lambda t e^{u(t)+\left(u''\left(t-\frac{1}{3}\right)\right)^2}\left(1+(u'(t))^2+\left(u\left(t-\frac{1}{2}\right)\right)^2+\left(u''\left(t-\frac{1}{4}\right)\right)^2\right)=0,\ \ t\in[0,1], \quad (24)$$

with the initial condition

$$u(t)=\psi(t), \quad t\in\left[-\frac{1}{2},0\right], \quad (25)$$

with $\psi(t)=H(-t)\cdot(1-\cos t)$, and one of the four BCs (22).

For example we choose $j = 3$, so that the functional BC is

$$u'''(1) = \lambda B[u], \tag{26}$$

where we fix

$$B[u] = \frac{1}{1 + (u(\frac{1}{2}))^2} + \int_{-\frac{1}{2}}^{1} t^3 (u''(t))^2 \, dt.$$

Thus the function $\widehat{\psi}$ is given by

$$\widehat{\psi}(t) = \begin{cases} 1 - \cos t, & -\frac{1}{2} \le t \le 0, \\ \frac{1}{2} t^2, & 0 < t \le 1. \end{cases}$$

Now choose $\rho \in (0, +\infty)$. We may take

$$\underline{\eta}_\rho(t) = \frac{1}{1 + \rho^2}, \ \underline{\delta}_\rho(t) = t.$$

Therefore we have

$$\sup_{t\in[0,1]}\left\{\frac{\frac{1}{2}t^2}{1+\rho^2}+\int_0^1 k(t,s)s\,ds\right\}\geq\frac{1}{6(1+\rho^2)}>0,$$

which implies that (9) is satisfied for every $\rho\in(0,+\infty)$.

As a consequence of Theorem 7 we obtain uncountably many pairs of solutions and parameters (u_ρ,λ_ρ) for the FBVP (24)–(25)–(26).

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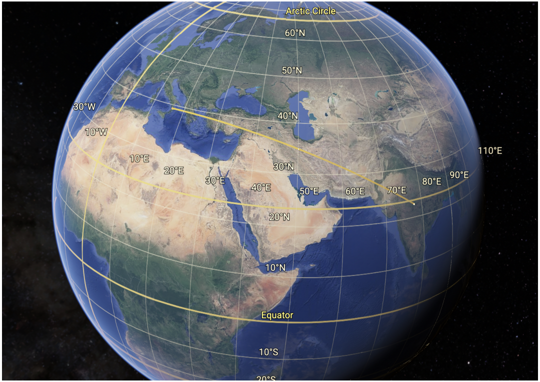

Thank you very much for your attention!